Foundations of Applied Mathematics
Volume 0

ACME Warmup Readings

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1 The Complex Numbers

For every complex problem there is an answer that is clear, simple, and wrong.
—H. L. Mencken

In this chapter we briefly review the fundamental properties of the field of complex numbers.

1.1 Complex Numbers

1.1.1 Basics of Complex Numbers

Definition 1.1.1. *Let i be a formal symbol (representing a square root of -1). Let $\mathbb{C}$ denote the set*

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$

Elements of $\mathbb{C}$ are called complex numbers*. We define addition of complex numbers by*

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and we define multiplication of complex numbers by

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Example 1.1.2. Complex numbers of the form $a+0i$ are usually written just as a, and those of the form $0 + bi$ are usually written just as bi.

We verify that i has the expected property:

$$i^2 = (0 + 1i)^2 = (0 + 1i)(0 + 1i) = (0 - 1) + (0 + 0)i = -1$$

Definition 1.1.3. *For any $z = a + bi \in \mathbb{C}$ we define the* complex conjugate *of z to be $\bar{z} = a - bi$, and we define the* modulus *(sometimes also called the* norm*) of z to* *be* $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$*. We also define the real part* $\Re(z) = a$*, and the imaginary part* $\Im(z) = b$*.*

Note that $|z| \in \mathbb{R}$ for any $z \in \mathbb{C}$.

Proposition 1.1.4. *Addition and multiplication of complex numbers satisfy the following properties. For any* $z, w, v \in \mathbb{C}$ *we have*

(*i*) *Associativity of addition:* $(v + w) + z = v + (w + z)$.

(*ii*) *Commutativity of addition:* $z + w = w + z$.

(*iii*) *Associativity of multiplication:* $(vw)z = v(wz)$.

(*iv*) *Commutativity of multiplication:* $zw = wz$.

(*v*) *Distributivity:* $v(w + z) = vw + vz$.

(*vi*) *Additive identity:* $0 + z = z = z + 0$.

(*vii*) *Multiplicative identity:* $1 \cdot z = z = z \cdot 1$.

(*viii*) *Additive inverses: if* $z = a + bi$*, then* $-z = -a - bi$ *satisfies* $z + (-z) = 0$.

(*ix*) *Multiplicative inverses: If* $z = a + bi \neq 0$ *then* $|z|^{-2} \in \mathbb{R}$ *and so*

$$z^{-1} = \bar{z}|z|^{-2} = \frac{a - bi}{a^2 + b^2}.$$

Proof. All of the properties are straightforward algebraic manipulations. We give one example and leave the rest to the reader.

For (ix) first note that since $z \neq 0$ we have $|z|^2 = a^2 + b^2 \neq 0$, so its multiplicative inverse $(a^2 + b^2)^{-1}$ is also in $\mathbb{R}$. We have

$$z\left(\bar{z}|z|^{-2}\right) = z\bar{z}(z\bar{z})^{-1} = 1,$$

so $\left(\bar{z}|z|^{-2}\right)$ is the multiplicative inverse to z. □

1.1.2 Euler's Formula and Graphical Representation

Euler's Formula

For any $z \in \mathbb{C}$ we define the exponential e^z using the Taylor series

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

One of the most important identities for complex numbers is *Euler's formula* (see Proposition **??**).

$$e^{it} = \cos(t) + i\sin(t) \tag{1.1}$$

for all $t \in \mathbb{C}$.

As a consequence of Euler's formula, we have *De Moivre's formula*

$$(\cos(t) + i\sin(t))^n = (e^{it})^n = e^{int} = \cos(nt) + i\sin(nt). \tag{1.2}$$

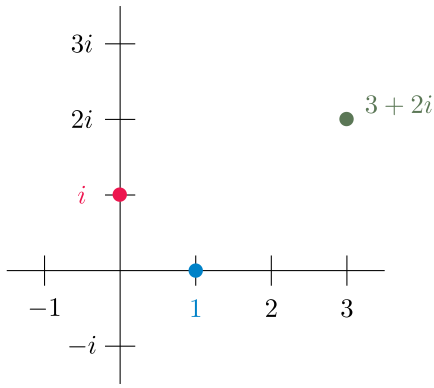

Figure 1.1: A complex number $x+iy$ with $x, y \in \mathbb{R}$ is usually represented graphically in the plane as the point (x, y). This figure shows the graphical representation of the complex numbers i, 1 and $3+2i$.

Graphical Representation

The complex numbers have a very useful graphical representation as points in the plane, where we associate the complex number $z = a+bi$ with the point $(a, b) \in \mathbb{R}^2$. In this representation real numbers lie along the x-axis and imaginary numbers lie along the y-axis. The modulus $|z|$ of z is the distance from the origin to z in the plane and the complex conjugate $\bar{z}$ is the image of z under a reflection through the x-axis.

Addition of complex numbers is just the same as vector addition in the plane; so geometrically, the complex number $z+w$ is the point in the plane corresponding to the far corner of the parallelogram whose other corners are 0, z, and w.

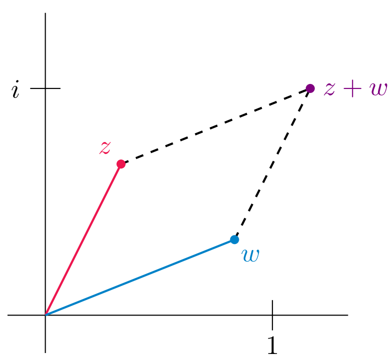

Figure 1.2: Graphical representation of complex addition. Thinking of complex numbers $z = a+bi$ and $w = c+di$ as the points in the plane (a, b) and (c, d), respectively, their sum $z+w = (a+c)+(b+d)i$ corresponds to the usual vector sum $(a, b)+(c, d) = (a+c, b+d)$ in the plane.

We can represent any point in the plane in polar form as $z = r(\cos(\theta)+i\sin(\theta))$

for some $\theta \in [0, 2\pi)$ and some $r \in \mathbb{R}$ with $r \geq 0$. Combining this with Euler's formula means that we can write every complex number in the form $z = re^{i\theta}$. In this form we have

$$|z| = |re^{i\theta}| = |r(\cos(\theta) + i\sin(\theta))| = r \qquad \text{and} \qquad \bar{z} = r(\cos(\theta) - i\sin(\theta)) = re^{-i\theta}.$$

We define the *sign* of $z = re^{i\theta} \in \mathbb{C}$ to be

$$\operatorname{sign}(z) = \begin{cases} e^{i\theta} = \frac{z}{|z|} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0. \end{cases} \tag{1.3}$$

We can use the polar form to get a geometric interpretation of multiplication of complex numbers. If $z = re^{it}$ and $w = \rho e^{is}$, then

$$wz = r\rho e^{i(t+s)} = |z||w|(\cos(t+s) + i\sin(t+s)).$$

Multiplication of two complex numbers in polar form multiplies the moduli and adds the angles.

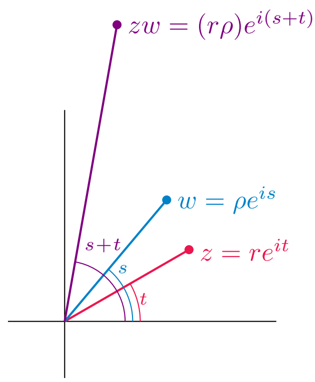

Figure 1.3: Complex multiplication adds the polar angles $(s + t)$ and multiplies the moduli $(r\rho)$.

Similarly, $z^{-1} = \bar{z}|z|^{-2} = re^{-it}r^{-2} = r^{-1}e^{-it}$, so the multiplicative inverse changes the sign of the angle $(t \mapsto -t)$ and inverts the modulus $(r \mapsto r^{-1})$. But the complex conjugate leaves the modulus unchanged and changes the sign of the angle.

1.1.3 Roots of Unity

Definition 1.1.5. *For $n \in \mathbb{Z}^+$ an* nth root of unity *is any solution to the equation $z^n = 1$ in $\mathbb{C}$. The complex number $\omega_n = e^{2\pi i/n}$ is called the* primitive nth root of unity.

By the fundamental theorem of algebra (or rather its corollary, Theorem **??**) there are exactly n of these nth roots of unity in $\mathbb{C}$. Euler's formula tells us that

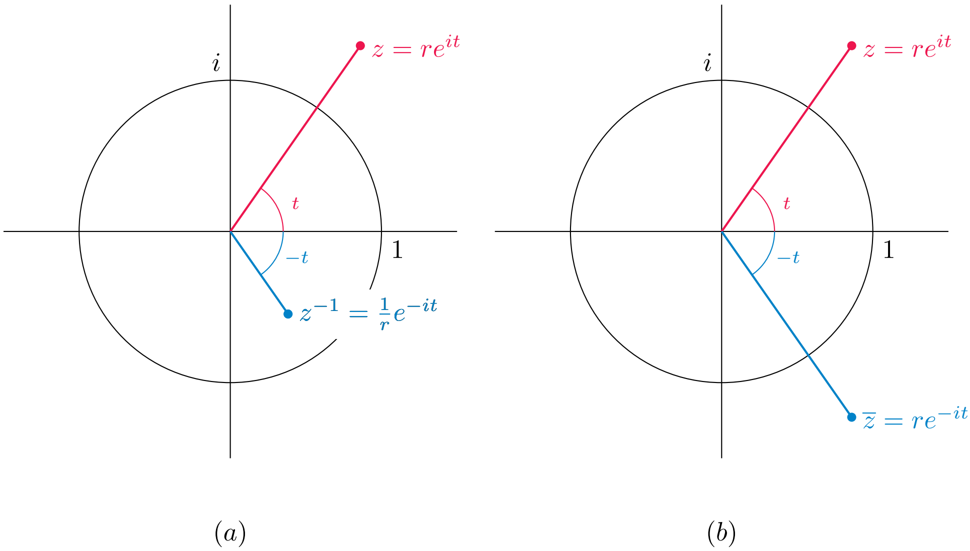

Figure 1.4: Graphical representation of multiplicative inverse (a), and complex conjugate (b). The multiplicative inverse of a complex number changes the sign of the polar angle and inverts the modulus. The complex conjugate also changes the sign of the polar angle, but leaves the modulus unchanged.

$\omega_n = \cos(2\pi/n) + i\sin(2\pi/n)$ is the point on the unit circle in the complex plane corresponding to an angle of $2\pi/n$ radians, and

$$\omega_n^k = e^{2\pi ik/n} = \cos(2\pi k/n) + i\sin(2\pi k/n).$$

Thus we have

$$\omega_n^n = e^{2\pi i} = 1,$$

so ω_n^k is a root of unity for every $k \in \mathbb{Z}$.

If $k' \equiv k \pmod{n}$, then $k' = k + mn$ for some $m \in \mathbb{Z}$, and thus

$$\omega_n^{k'} = \omega_n^{(k+mn)} = \omega_n^k(\omega_n^n)^m = \omega_n^k.$$

The nth roots of unity are uniformly distributed around the unit circle, so their average is 0. The next proposition makes that precise.

Proposition 1.1.6. *For any $n \in \mathbb{Z}^+$ and any $k \in \mathbb{Z}$ we have*

$$\frac{1}{n}\sum_{\ell=0}^{n-1}\omega_n^{k\ell} = \frac{1}{n}\sum_{\ell=0}^{n-1}e^{2\pi ik\ell/n} = \begin{cases} 0 & k \not\equiv 0 \pmod{n} \\ 1 & k \equiv 0 \pmod{n}. \end{cases} \tag{1.4}$$

Proof. The sum $\sum_{\ell=0}^{n-1}(\omega_n^k)^\ell$ is a geometric series, so if $k \not\equiv 0 \pmod{n}$ we have

$$\sum_{\ell=0}^{n-1}\omega_n^{k\ell} = \frac{(\omega_n^k)^n - 1}{\omega_n^k - 1} = \frac{(\omega_n^n)^k - 1}{\omega_n^k - 1} = 0.$$

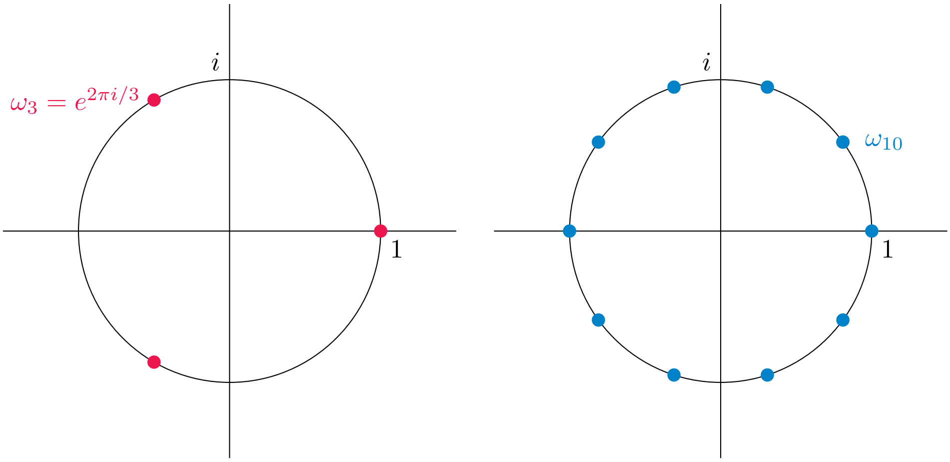

Figure 1.5: Plots of all the 3rd (on the left) and 10th (on the right) roots of unity. The roots are uniformly distributed around the unit circle, so their sum is 0.

But if $k \equiv 0 \pmod{n}$ then

$$\frac{1}{n}\sum_{\ell=0}^{n-1}\omega_n^{k\ell} = \frac{1}{n}\sum_{\ell=0}^{n-1} 1 = 1. \quad \square$$

We conclude this section with a simple observation that turns out to be very powerful. The proof is immediate.

Proposition 1.1.7. *For any divisor d of n and any $k \in \mathbb{Z}$, we have*

$$\omega_n^{kd} = \omega_{n/d}^{k}. \tag{1.5}$$

Vista 1.1.8. The relation (1.5) is the foundation of the fast Fourier transform (FFT). We discuss the FFT in Volume 2.

Exercises

1.1 Express each of the following complex numbers in polar form (i.e., in the form $re^{i\theta}$) and plot them in the complex plane.

(a) $z = 2 + 2\sqrt{3}i$

(b) $w = -2 + 2i$

(c) $z + w$

(d) zw

(e) z/w

1.2 For any $z, w \in \mathbb{C}$ prove that

(a) $\overline{z+w} = \bar{z} + \bar{w}$

(b) $\overline{z-w} = \bar{z} - \bar{w}$

(c) $\overline{zw} = \bar{z}\bar{w}$

(d) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

1.3 Express each of the following as a complex number of the form $a + ib$, where $a, b \in \mathbb{R}$. Is the solution unique? Justify your answer.

(a) $\sqrt{i}$

(b) $\sqrt{1+i}$

(c) $\sqrt{\sqrt{-i}}$

1.4 If $|z| = 1$, and $a, b \in \mathbb{C}$, prove that

$$\left|\frac{az+b}{\bar{b}z+\bar{a}}\right| = 1.$$

Hint: remember that $|w|^2 = w\bar{w}$, and $\overline{rw+s} = \bar{r}\bar{w} + \bar{s}$.

1.5 Show that

$$\frac{e^{i(b-a)}}{b-a} + \frac{e^{i(a-b)}}{a-b} = \frac{2i\sin(a-b)}{a-b}.$$

1.6 Find all the complex numbers ζ satisfying the relation $\zeta^2+\zeta+1 = 0$ as follows:

(a) First determine how many solutions exist.

(b) Show that any such ζ must satisfy $\zeta^3 = 1$ and $\zeta \neq 1$.

(c) Use the previous step to find the polar form of all the solutions.

(d) Now solve the problem using the rectangular form by writing $\zeta = x + iy$ and computing $(x+iy)^2+(x+iy)+1$. Setting both the real and imaginary parts to zero gives two equations in two unknowns, whose solutions give the required values of x and y.

(e) Show that the answers you got in polar form agree with the answers you got in rectangular form.

1.7 Show for all $k \in \mathbb{Z}$, with $k \neq 0$, that

$$\int_0^{2\pi} e^{ikt}\,dt = 0.$$

A The Greek Alphabet

I fear the Greeks even when they bring gifts.
—Virgil

Capital	Lower	Variant	Name
A	α		Alpha
B	β		Beta
Γ	γ		Gamma
Δ	δ		Delta
E	ϵ	ε	Epsilon
Z	ζ		Zeta
H	η		Eta
Θ	θ	ϑ	Theta
I	ι		Iota
K	κ	$\varkappa$	Kappa
Λ	λ		Lambda
M	μ		Mu
N	ν		Nu
Ξ	ξ		Xi
O	o		Omicron
Π	π	ϖ	Pi
P	ρ	ϱ	Rho
Σ	σ	ς	Sigma
T	τ		Tau
Υ	υ		Upsilon
Φ	ϕ	φ	Phi
X	χ		Chi
Ψ	ψ		Psi
Ω	ω		Omega