1 Timber Harvesting

Lab Objective: *Our main goal is to introduce solving problems linear in the control through a simple tree harvesting simulation.*

Introduction

Timber harvesting is a much needed part of business logistics management. Timber is the beginning of the supply chain because it provides the raw material for many housing, construction, and paper products. Now imagine you are the owner of a timber farm which, due to environmental regulations, can only harvest at most a fixed percentage of its trees, which must then be replanted. We will assume the farm operates at this fixed production rate and we will let $x(t)$ be the amount of raw timber produced at time t.

We will make the following assumptions: - growth and death rates of individual trees need not be considered - tree age (assume harvest percentage level is low enough) - amount of timber is based solely on the size of the farm (i.e. number of trees)

Outlining the Problem

In this problem, we will also assume that once the timber has been processed, it is immediately sold. The money can either be kept as profit or reinvested in the farm by purchasing land and labor for further tree growth. You, the owner of the farm, wish to find the reinvestment schedule that will maximize profit over a fixed time interval. We will let the control be $u(t)$ which represents the percentage of timber revenue reinvested in the time frame. Since reinvestment in harvesting leads to more tree growth which in turn leads to more timber production, we have $x'(t) = kx(t)u(t)$, where k is the return constant, which takes into account the average cost of labor and land. If p is the market price of a unit of timber, then the profit at time t is $px(t)(1 - u(t))$, and the total profit is

$$p \int_0^T x(t)[1 - u(t)]\, dt$$

However, we also want to take into account money which could be gained through interest on profit. This means that profit earned in year one could be placed into an interest bearing account, while profit from the end of the time period could not. In other words, money from earlier in the time period is, in some sense, more valuable. In economics, this is referred to as present-value. Now we will let r be the interest rate over the period, which we will assume is fixed, so our profit, adjusted for interest, is

$$p \int_0^T e^{-rt} x(t)[1 - u(t)]\, dt$$

The exponential term in the integral is called a discount term. Notice that the function e^{-rt} is a decreasing function of time, which encourages money to be invested at the beginning of the interval and future profit is discounted at a rate r. Notice that the constant p does not affect how the integral is maximized so we do not need to include. Therefore our optimal control problem becomes

$$\max_u \int_0^T e^{-rt} x(t)[1 - u(t)]\, dt \tag{1.1}$$

subject to the constraints

$$\text{x'(t)} = \text{kx(t)u(t)} \qquad 2005/06/28ver:1.3s$$

$$x(0) = x_0 > 0 \tag{1.3}$$

$$0 \leq u(t) \leq 1 \tag{1.4}$$

We can then solve for the Hamiltonian which is given by

$$H = e^{-rt} x(1 - u) + kxu\alpha \tag{1.5}$$

By developing the optimality system as usual, we find

$$\alpha' = u(e^{-rt} - k\alpha) - e^{-rt} \tag{1.6}$$

$$\phi = \frac{\delta H}{\delta u} = x(k\alpha - \mathbf{e}{-}rt) \tag{1.7}$$

As $x(0) = x_0 > 0$, $k > 0$, and $u \geq 0$ for all t, it follows that $x'(t) = kxu \geq 0$ and $x(t) > 0$ for all t. Suppose $\phi = 0$ over some interval. As x is strictly positive, this can occur if and only if $\alpha(t) = 1/ke^{-rt}$ over some interval. Then, $\alpha'(t) = -r/ke^{-rt}$. However, if we use the adjoint equation, we instead find $\alpha'(t) = -e^{-rt}$. If $k \neq r$, this is clearly a contradiction. If $k = r$, then it follows that $\alpha(t) = 1/ke^{-rt}$ for the remainder of the time interval. This contradicts $\alpha(T) = 0$. Thus, $\phi = 0$ cannot be sustained over an interval, and the optimal control is bang-bang.

Creating a Numerical Solver

We iteratively solve for our control u. In each iteration we solve our state equations and our costate equations numerically, then use those to find our new control. Lastly, we check to see if our control has converged. To solve each set of differential equations, we will use the RK4 solver from a previous lab with one minor adjustment. Our state equations depend on u, and our costate equations depend on our state equations. Therefore, we will pass another parameter into the function that RK4 takes in that will index the arrays our equations depend on.

```python
# Dependencies for this lab's code:
import numpy as np
from matplotlib import pyplot as plt

#Code from RK4 Lab with minor edits

```

```python
def RK4(x0,k,r,T,N=1000,delta=0.001):
    """ Use the RK4 method to compute an approximate solution
    to the ODE y' = f(t, y) at n equispaced parameter values from t0 to t
    with initial conditions y(t0) = y0.

    This function returns an array Y of shape (n,) if
    y is a constant or an array of size 1.
    It returns an array of shape (n, y.size) otherwise.
    In either case, Y[i] is the approximate value of y at
    the i'th value of np.linspace(t0, tf, n).
    """
    t = np.linspace(0,T,N+1)
    h = T/N
    h2 = h/2

    x = np.zeros(N+1)
    alpha = np.zeros(N+1)
    u = np.zeros(N+1)

    x[0] = x0

    for i in range(N):
        k1 = k*u[i]*x[i]
        k2 = k*1/2*(u[i]+u[i+1])*(x[i]+h2*k1)
        k3 = k*1/2*(u[i]+u[i+1])*(x[i]+h2*k2)
        k4 = k*u[i+1]*(x[i]+h*k3)
        x[i+1] = x[i]+h/6*(k1+2*k2+2*k3+k4)

    # you will need to do something similar for (1.6) but solve for alpha ↩
        backwards in time and then use (1.7) to update u

    # use the following for your stopping criteria and stop once test > 0
    temp1 = delta*np.sum(np.abs(u)) - np.sum(np.abs(oldu - u))
    temp2 = delta*np.sum(np.abs(x)) - np.sum(np.abs(oldx - x))
    temp3 = delta*np.sum(np.abs(alpha)) - np.sum(np.abs(oldalpha - alpha))
    test = np.min([temp1, np.min([temp2, temp3])])
```

Problem 1. Write a function that takes as input scalars x_0, k, r, and a final time T and solves the optimal control problem stated above using the RK4 method described above. The function will return the time-step and the values of x and u at the specific time-steps.

Problem 2. Using your function from problem 1, plot time vs. timber production (x) and plot time vs. reinvestment percentage (u) for the following values: $x_0 = 100$, $k = 1$, $r = 0$, and $T = 5$.

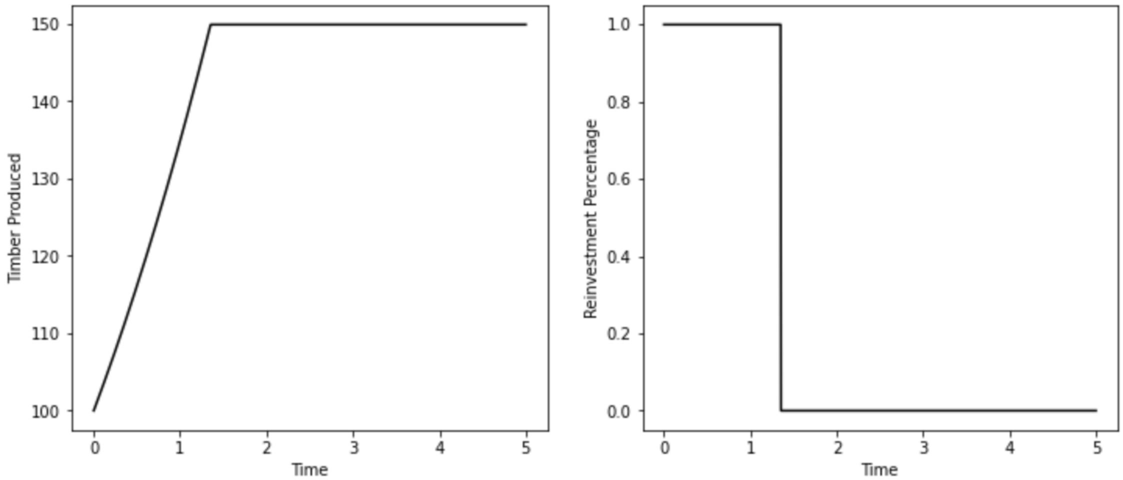

Figure 1.1: The solution for problem 1

.

In the graph you just plotted, you will notice that in the control the graph is continuous and there is an abrupt shift at $t = 4$. We knew this problem was bang-bang and the vertical portion at $t = 4$ represents the switching point. We can also notice that $x(t)$ is exponential for $0 \leq t \leq 4$ and constant for $4 \leq t \leq 5$, as we would expect. This scenario demonstrates that the optimal reinvestment strategy is to reinvest all timber revenue for the first four-fifths of the time interval, allowing the farm to grow. Towards the end of the time interval, all revenue should instead be kept as profit.

Problem 3. Plot the same graphs that you did in problem 2, but instead for the following values: $x_0 = 100$, $k = 0.3$, $r = 0.05$, and $T = 5$. At what time does the switching point occur?

Problem 4. Now use the same parameters as in problem 3, but vary the initial value of the timber production capacity (x_0). You should try a smaller value, a slightly larger value, and fairly larger value for x_0. What do you notice about the optimal controls (are they the same or are they different)?