

1 Visualizing Complex-valued Functions

Lab Objective: *Functions that map from the complex plane into the complex plane are difficult to fully visualize because the domain and range are both 2-dimensional. However, such functions can be visualized at the expense of partial information. In this lab we present methods for analyzing complex-valued functions visually, including locating their zeros and poles in the complex plane. We recommend completing the exercises in a Jupyter Notebook.*

Representations of Complex Numbers

A complex number $z = x + iy$ can be written in *polar coordinates* as $z = re^{i\theta}$ where

- $r = |z| = \sqrt{x^2 + y^2}$ is the *magnitude* of z , and
- $\theta = \arg(z) = \arctan(y/x)$ is the *argument* of z , the angle in radians between z and 0.

Conversely, Euler's formula is the relation $re^{i\theta} = r \cos(\theta) + ir \sin(\theta)$. Then setting $re^{i\theta} = x + iy$ and equating real and imaginary parts yields the equations $x = r \cos(\theta)$ and $y = r \sin(\theta)$.

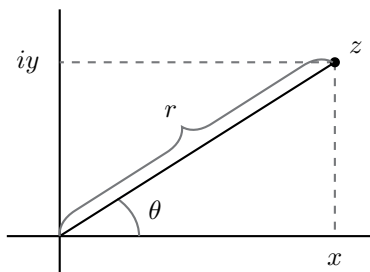


Figure 1.1: The complex number z can be represented in Cartesian coordinates as $z = x + iy$ and in polar coordinates as $z = re^{i\theta}$, when θ is in radians.

NumPy makes it easy to work with complex numbers and convert between coordinate systems. The function `np.angle()` returns the argument θ of a complex number (between $-\pi$ and π) and `np.abs()` (or `np.absolute()`) returns the magnitude r . These functions also operate element-wise on NumPy arrays.

```

>>> import numpy as np

>>> z = 2 - 2*1j                                # 1j is the imaginary unit i = sqrt(-1).
>>> r, theta = np.abs(z), np.angle(z)
>>> print(r, theta)                             # The angle is between -pi and pi.
2.82842712475 -0.785398163397

# Check that z = r * e^(i*theta)
>>> np.isclose(z, r*np.exp(1j*theta))
True

# These function also work on entire arrays.
>>> np.abs(np.arange(5) + 2j*np.arange(5))
array([ 0.          ,  2.23606798,  4.47213595,  6.70820393,  8.94427191])

```

Complex Functions

A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is called a *complex-valued function*. Visualizing f is difficult because \mathbb{C} has 2 real dimensions, so the graph of f should be 4-dimensional. However, since it is possible to visualize 3-dimensional objects, f can be visualized by ignoring one dimension. There are two main strategies for doing this: assign a color to each point $z \in \mathbb{C}$ corresponding to either the argument θ of $f(z)$, or to the magnitude r of $f(z)$. The graph that uses the argument is called a *complex color wheel graph*. Figure 1.2 displays the identity function $f(z) = z$ using these two methods.

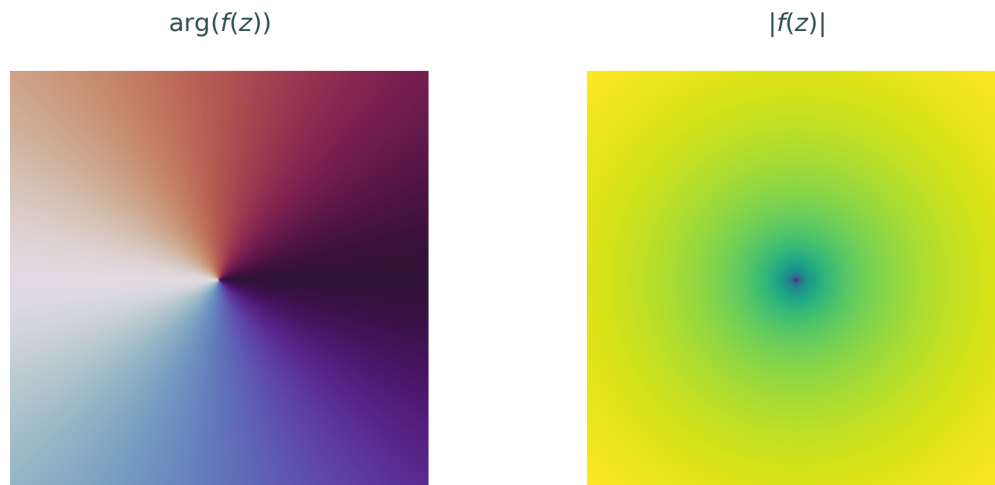


Figure 1.2: The identity function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(z) = z$. On the left, the color at each point z represents the angle $\theta = \arg(f(z))$. As θ goes from $-\pi$ to π , the colors cycle smoothly counterclockwise from white to blue to red and back to white (this colormap is called "twilight"). On the right, the color represents the magnitude $r = |f(z)|$. The further a point is from the origin, the greater its magnitude (the colormap is the default, "viridis").

The plots in Figure 1.2 use Cartesian coordinates in the domain and polar coordinates in the codomain. The procedure for plotting in this way is fairly simple. Begin by creating a grid of complex numbers: create the real and imaginary parts separately, then use `np.meshgrid()` to turn them into a single array of complex numbers. Pass this array to the function f , compute the angle and argument of the resulting array, and plot them using `plt.pcolormesh()`. The following code sets up the complex domain grid.

```
>>> x = np.linspace(-1, 1, 400)      # Real domain.
>>> y = np.linspace(-1, 1, 400)      # Imaginary domain.
>>> X, Y = np.meshgrid(x, y)         # Make grid matrices.
>>> Z = X + 1j*Y                     # Combine the grids into a complex array.
```

Visualizing the argument and the magnitude separately provides different perspectives of the function f . The angle plot is generally more useful for visualizing function behavior, though the magnitude plot often makes it easy to spot important points such as zeros and poles.

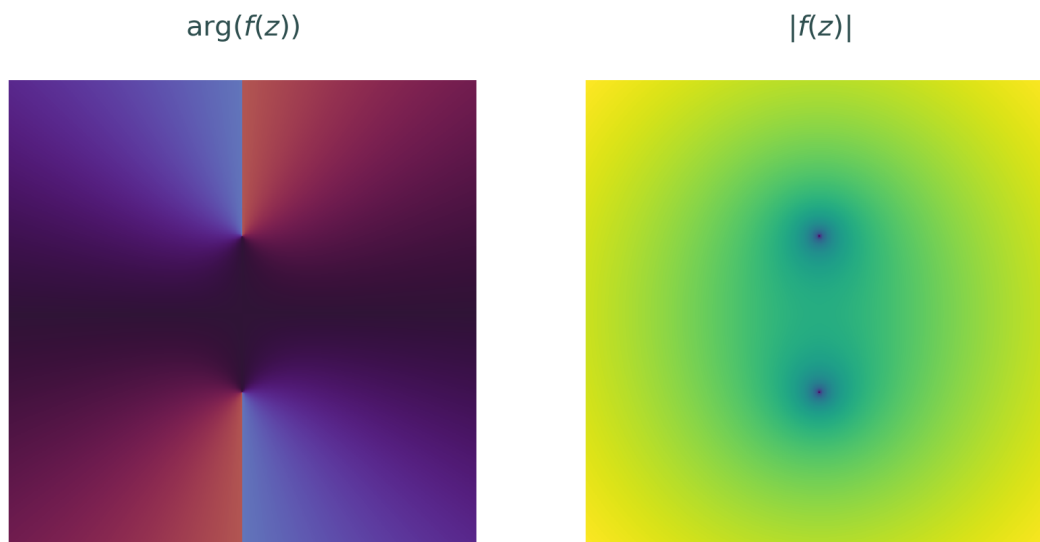


Figure 1.3: Plots of $f(z) = \sqrt{z^2 + 1}$ on $\{x + iy \mid x, y \in [-3, 3]\}$. Notice how a discontinuity is clearly visible in the angle plot on the left, but disappears from the magnitude plot on the right.

Problem 1. Write a function that accepts a function $f : \mathbb{C} \rightarrow \mathbb{C}$, bounds $[r_{\min}, r_{\max}, i_{\min}, i_{\max}]$ for the domain, an integer `res` that determines the resolution of the plot, and a string to set the figure title. Plot $\arg(f(z))$ and $|f(z)|$ on an equally-spaced `res` \times `res` grid over the domain $\{x + iy \mid x \in [r_{\min}, r_{\max}], y \in [i_{\min}, i_{\max}]\}$ in separate subplots.

1. For $\arg(f(z))$, set the `plt.pcolormesh()` keyword arguments `vmin` and `vmax` to $-\pi$ and π , respectively. This forces the color spectrum to work well with `np.angle()`. Use the colormap `"twilight"`, which starts and ends white, so that the color is the same for $-\pi$ and π .

2. For $|f(z)|$, set `norm=matplotlib.colors.LogNorm()` in `plt.pcolormesh()` so that the color scale is logarithmic. Use a sequential colormap like `"viridis"` or `"magma"`.
3. Set the aspect ratio to `"equal"` in each plot. Give each subplot a title, and set the overall figure title with the given input string.

Use your function to visualize $f(z) = z$ on $\{x + iy \mid x, y \in [-1, 1]\}$ and $f(z) = \sqrt{z^2 + 1}$ on $\{x + iy \mid x, y \in [-3, 3]\}$. Compare the resulting plots to Figures 1.2 and 1.3, respectively.

Interpreting Complex Plots

Plots of a complex function can be used to quickly identify important points in the function's domain.

Zeros

A complex number z_0 is called a *zero* of the complex-valued function f if $f(z_0) = 0$. The *multiplicity* or *order* of z_0 is the largest integer n such that f can be written as $f(z) = (z - z_0)^n g(z)$ where $g(z_0) \neq 0$. In other words, f has a zero of order n at z_0 if the Taylor series of f centered at z_0 can be written as

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k, \quad a_n \neq 0.$$

Angle and magnitude plots make it easy to locate a function's zeros and to determine their multiplicities.

Problem 2. Use your function from Problem 1 to plot the following functions on the domain $\{x + iy \mid x, y \in [-1, 1]\}$.

- $f(z) = z^n$ for $n = 2, 3, 4$.
- $f(z) = z^3 - iz^4 - 3z^6$. Compare the resulting plots to Figure 1.4.

Use a Markdown cell to write a sentence or two about how the zeros of a function and their multiplicity appear in angle and magnitude plots.

Problem 2 shows that in an angle plot of $f(z) = z^n$, the colors cycle n times counterclockwise around 0. This is explained by looking at z^n in polar coordinates,

$$z^n = (re^{i\theta})^n = r^n e^{i(n\theta)}.$$

Multiplying θ by a number greater than 1 compresses the graph along the " θ -axis" by a factor of n . In other words, the output angle repeats itself n times in one cycle of θ . This is similar to taking a scalar-valued function $f: \mathbb{R} \rightarrow \mathbb{R}$ and replacing $f(x)$ with $f(nx)$.

Problem 2 also shows that the plot of $f(z) = z^3 - iz^4 - 3z^6$ looks very similar to the plot of $f(z) = z^3$ near the origin. This is because when z is close to the origin, z^4 and z^6 are much smaller in magnitude than z^3 , and so the behavior of z^3 dominates the function. In terms of the Taylor series centered at $z_0 = 0$, the quantity $|z - z_0|^{n+k}$ is much smaller than $|z - z_0|^n$ for z close to z_0 , and so the function behaves similar to $a_n(z - z_0)^n$.

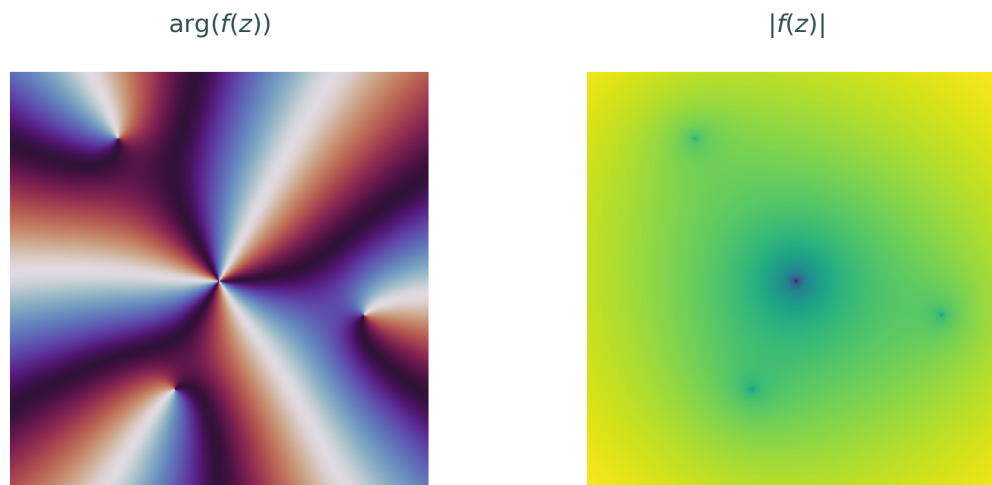


Figure 1.4: The angle plot of $f(z) = z^3 - iz^4 - 3z^6$ on $\{x + iy \mid x, y \in [-1, 1]\}$. The angle plot shows that $f(z)$ has a zero of order 3 at the origin and 3 distinct zeros of order 1 scattered around the origin. The magnitude plot makes it easier to pinpoint the location of the zeros.

Poles

A complex number z_0 is called a *pole* of the complex-valued function f if f can be written as $f(z) = g(z)/(z - z_0)$ where $g(z_0) \neq 0$. From this definition it is easy to see that $\lim_{z \rightarrow z_0} |f(z)| = \infty$, but knowing that $\lim_{z \rightarrow z_1} |f(z)| = \infty$ is not enough information to conclude that z_1 is a pole of f .

The *order* of z_0 is the largest integer n such that f can be written as $f(z) = g(z)/(z - z_0)^n$ with $g(z_0) \neq 0$. In other words, f has a pole of order n at z_0 if its Laurent series on a punctured neighborhood of z_0 can be written as

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k, \quad a_{-n} \neq 0.$$

Problem 3. Plot the following functions on domains that show all of its zeros and/or poles.

- $f(z) = z^{-n}$ for $n = 1, 2, 3$.
- $f(z) = z^2 + iz^{-1} + z^{-3}$.

Use a Markdown cell to write a sentence or two about how the poles of a function appear in angle and magnitude plots. How can you tell the multiplicity of the poles from the plot?

Problem 3 shows that in angle plot of z^{-1} , the colors cycle n times clockwise around 0, as opposed to the counter-clockwise rotations seen around roots. Again, this can be explained by looking at the polar representation,

$$z^{-n} = (re^{i\theta})^{-n} = r^{-n}e^{i(-n\theta)}.$$

The minus sign on the θ reverses the direction of the colors, and the n makes them cycle n times.

From Problem 3 it is also clear that $f(z) = z^2 + iz^{-1} + z^{-3}$ behaves similarly to z^{-3} for z near the pole at $z_0 = 0$. Since $|z - z_0|^{-n+k}$ is much smaller than $|z - z_0|^{-n}$ when $|z - z_0|$ is small, near z_0 the function behaves like $a_{-n}(z - z_0)^{-n}$. This is why the order of a pole can be estimated by counting the number of times the colors circle a point in the clockwise direction.

Counting Zeros and Poles

The *Fundamental Theorem of Algebra* states that a polynomial f with highest degree n has exactly n zeros, counting multiplicity. For example, $f(z) = z^2 + 1$ has two zeros, and $f(z) = (z - i)^3$ has three zeros, all at $z_0 = i$ (that is, $z_0 = i$ is a zero with multiplicity 3).

The number of poles of function can also be apparent if it can be written as a quotient of polynomials. For example, $f(z) = z/(z+i)(z-i)^2$ has one zero and three poles, counting multiplicity.

Problem 4. Plot the following functions and count the number and order of their zeros and poles. Adjust the bounds of each plot until you have found all zeros and poles.

- $f(z) = -4z^5 + 2z^4 - 2z^3 - 4z^2 + 4z - 4$
- $f(z) = z^7 + 6z^6 - 131z^5 - 419z^4 + 4906z^3 - 131z^2 - 420z + 4900$
- $f(z) = \frac{16z^4 + 32z^3 + 32z^2 + 16z + 4}{16z^4 - 16z^3 + 5z^2}$

It is usually fairly easy to see how many zeros or poles a polynomial or quotient of polynomials has. However, it can be much more difficult to know how many zeros or poles a different function may or may not have without visualizing it.

Problem 5. Plot the following functions on the domain $\{x + iy \mid x, y \in [-8, 8]\}$. Explain carefully in a Markdown cell what each graph reveals about the function and why the function behaves that way.

- $f(z) = e^z$
- $f(z) = \tan(z)$

(Hint: use the polar coordinate representation to mathematically examine the magnitude and angle of each function.)

Essential Poles

A complex-valued function f has an *essential pole* at z_0 if its Laurent series in a punctured neighborhood of z_0 requires infinitely many terms with negative exponents. For example,

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2} \frac{1}{z^2} + \frac{1}{6} \frac{1}{z^3} + \dots$$

An essential pole can be thought of as a pole of order ∞ . Therefore, in an angle plot the colors cycle infinitely many times around an essential pole.

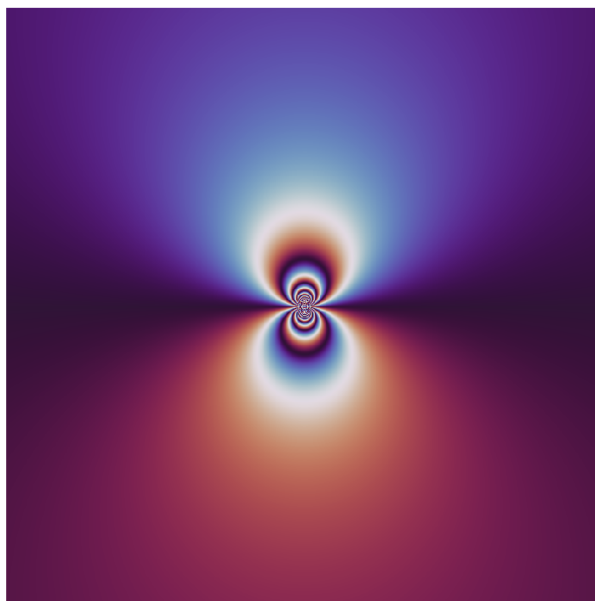


Figure 1.5: Angle plot of $f(z) = e^{1/z}$ on the domain $\{x + iy \mid x, y \in [-1, 1]\}$. The colors circle clockwise around the origin because it is a pole, not a zero. Because the pole is essential, the colors repeat infinitely many times.

ACHTUNG!

Often, color plots like the ones presented in this lab can be deceptive because of a bad choice of domain. Be careful to validate your observations mathematically.

Problem 6. For each of the following functions, plot the function on $\{x + iy \mid x, y \in [-1, 1]\}$ and describe what this view of the plot seems to imply about the function. Then plot the function on a domain that allows you to see the true nature of the roots and poles and describe how it is different from what the original plot implied. Use Markdown cells to write your answers.

- $f(z) = 100z^2 + z$
- $f(z) = \sin\left(\frac{1}{100z}\right)$.

(Hint: zoom way in.)