1 Predator-Prey and Weight Change Models

Lab Objective: *We introduce built-in methods for solving Initial Value Problems and apply the methods to two dynamical systems. The first system looks at the relationship between a predator and its prey. The second model is a weight change model based on thermodynamics and kinematics.*

ODE Solvers

Initial Value Problems (IVPs) are a systems of one or more ordinary differential equations (ODEs) with defined initial conditions. In some cases, these can be solved by hand, but in real life it is more practical to use numerical solvers. In the previous lab, you built your own numerical solvers. For this lab you will use the `odeint` solver from the `scipy.integrate` library.

`odeint` solves a system of ODEs given by $dy/dt = f(y,t), \quad y(t_0) = y_0$, where y can be a vector. The solver takes as parameters the callable function f, the initial condition y_0, and an array of time points t. Then `odeint` returns the array y with shape $(len(t), len(y_0))$, where each row gives the y values for one time point. The syntax for `odeint` is shown below, note this is the same syntax as the solvers built in the previous lab.

```
from scipy.integrate import odeint
sol = odeint(f, y0, t)
```

Assuming that f, $y0$, and t are previously defined (as explained above), *sol* is a vector containing the solution to the IVP and can be visualized by plotting each column of *sol* against the time domain or by plotting the columns against each other.

Predator-Prey Model

ODEs are commonly used to model relationships between predator and prey populations. For example, consider the populations of wolves (the predator) and rabbits (the prey) in Yellowstone National Park. Let $r(t)$ and $w(t)$ represent the rabbit and wolve populations respectively at time t, where the unit for time is years. We will make a few assumptions to simplify our model:

- In the absence of wolves, the rabbit population grows at a positive rate proportional to the current population. Thus when $w(t) = 0$, $dr/dt = \alpha r(t)$, where $\alpha > 0$.

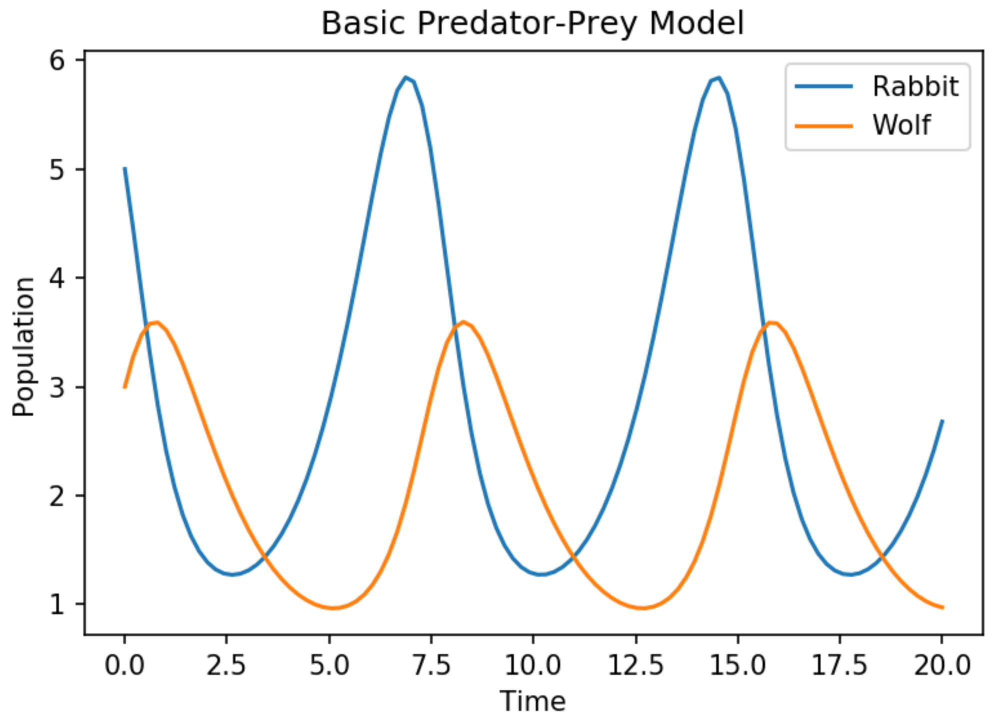

Figure 1.1: The solution to the system found in (1.1)

- In the absence of rabbits, the wolves die out. Thus when $r(t) = 0$, $dw/dt = -\delta w(t)$, where $\delta > 0$.
- The number of encounters between rabbits and wolves is proportional to the product of their populations. The wolf population grows proportional to the number of encounters by $\beta r(t)w(t)$ (where $\beta > 0$), and the rabbit population decreases proportional to the number of encounters by $-\gamma r(t)w(t)$ (where $\gamma > 0$).

This leads to the following system of ODEs:

$$\begin{aligned} \frac{dr}{dt} &= \alpha r - \beta r w = r(\alpha - \beta w) \\ \frac{dw}{dt} &= -\delta w + \gamma r w = w(-\delta + \gamma r) \end{aligned} \tag{1.1}$$

Problem 1. As mentioned above, the `odeint` solver requires a callable function representing the right hand side of the IVP. Define the function `predator_prey()` that accepts the current $r(t)$ and $w(t)$ values as a 1d array y, and the current time t, and returns the right hand side of (1.1) as a tuple. Use $\alpha = 1.0$, $\beta = 0.5$, $\delta = 0.75$, and $\gamma = 0.25$ as your growth parameters.

Problem 2. Use `odeint` to solve (1.1) with initial conditions $(r_0, w_0) = (5, 3)$ and time ranging from 0 to 20 years. Display the resulting rabbit and wolf populations over time (stored as columns in the output of `odeint`) on the same plot. Your graph should match the graph in figure 1.1.

Variations on the Predator-Prey

The Lotka-Volterra model

Reconsider (1.1). This representation of the predator-prey relationship is called the Lotka-Volterra predator-prey model and is typically given by

$$\frac{du}{dt} = \alpha u - \beta uv,$$
$$\frac{dv}{dt} = -\delta v + \gamma uv.$$

where u and v represent the prey and predator populations, respectively. Here α, β, δ, and γ are the same as before but now for an arbitrary prey and predator.

Let us look at the dynamics of this system. The equlibria (fixed points) of a system occur when the derivatives are zero, for our system this occurs at $(u, v) = (0, 0)$ and $(u, v) = (\frac{c}{d}, \frac{a}{b})$. Visualizing the phase portrait helps to give more insight into the dynamics of a system. We will do this by first nondimensionalzing our system to reduce the number of parameters. Let $U = \frac{\gamma}{\delta}u$, $V = \frac{\beta}{\alpha}v$, $\bar{t} = \alpha t$, and $\eta = \frac{\gamma}{\alpha}$. Substituting into the original ODEs we obtain the nondimensional system of equations

$$\begin{aligned}\frac{dU}{d\bar{t}} &= U(1 - V),\\ \frac{dV}{d\bar{t}} &= \eta V(U - 1).\end{aligned} \tag{1.2}$$

Problem 3. Similar to problem 1, define the function `Lotka_volterra()` that takes in the current predator and prey populations as a 1d array y and the current time as a float t and returns the right hand side of the system (1.2) with $\eta = 1/3$.

The following three lines of code plot the phase portrait of (1.2). For more documentation on quiver plots see https://matplotlib.org/2.0.0/api/_as_gen/matplotlib.axes.Axes.quiver.html.

```
Y1, Y2 = np.meshgrid(np.linspace(0, 4.5, 25), np.linspace(0, 4.5, 25))
dU, dV = Lotka_Volterra((Y1, Y2), 0)
Q = plt.quiver(Y1[::3, ::3], Y2[::3, ::3], U[::3, ::3], V[::3, ::3])
```

Using `odeint`, solve (1.2) with three different initial conditions $y_0 = (1/2, 1/3)$, $y_0 = (1/2, 3/4)$, and $y_0 = (1/16, 3/4)$ and time domain $t = [0, 13]$. Plot these three solutions on the same graph as the phase portrait and the equilibria $(0, 0)$ and $(1, 1)$.

Since your solutions are being plotted with the phase portrait, plot the two populations against each other (instead of both individually against time). Your plot should match 1.2.

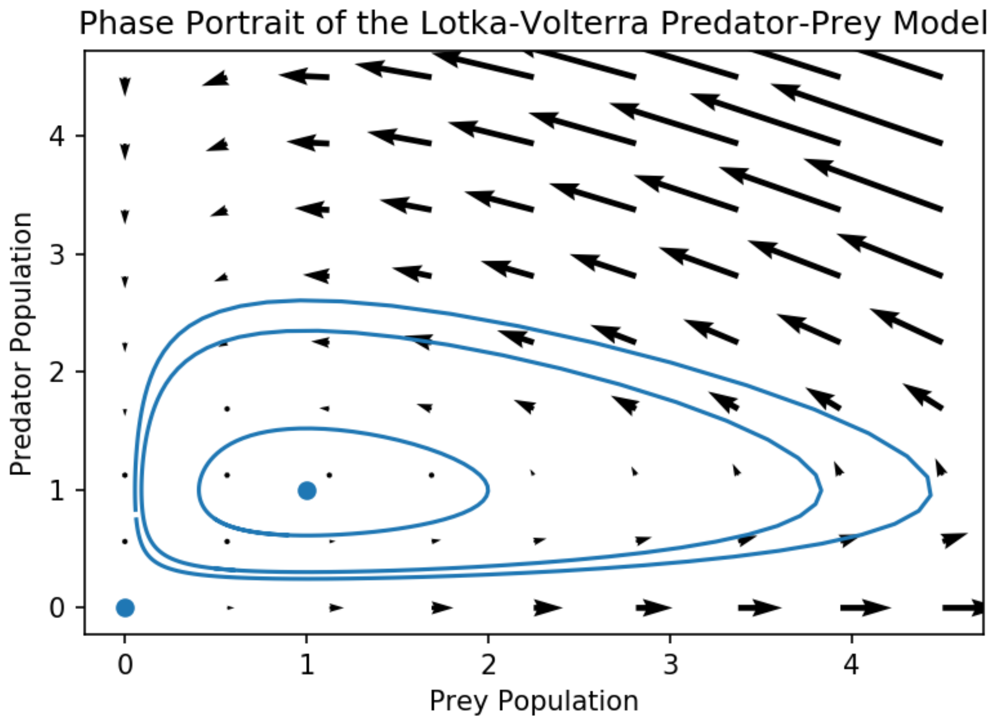

Figure 1.2: The phase portrait for the nondimensionalized Lotka-Volterra predator-prey equations with parameters $\eta = 1/3$.

The Logistic model

Notice that the Lotka-Volterra equations predict prey populations will grow exponentially in the absence of predators. The logistic predator-prey equations change this dynamic by adding a carrying capacity K to the prey population:

$$\frac{du}{dt} = \alpha u\left(1 - \frac{u}{K}\right) - \beta uv,$$
$$\frac{dv}{dt} = -\delta v + \gamma uv.$$

We can again do dimensional analysis on this system to simplify parameters. Let $U = \frac{u}{K}$, $V = \frac{\beta}{\alpha}v$, $\bar{t} = \alpha t$, $\eta = \frac{\gamma K}{\alpha}$, and $\rho = \frac{\delta}{\gamma K}$. Then the nondimensional logistic equations are

$$\begin{aligned}\frac{dU}{d\bar{t}} &= U(1 - U - V),\\ \frac{dV}{d\bar{t}} &= \eta V(U - \rho).\end{aligned} \tag{1.3}$$

Problem 4. Define a new function `Logistic_Model()` that takes in the current predator and prey populations y and the current time t and returns the right hand side of (1.3) as a tuple.

Use `odeint` to compute solutions (U, V) of (1.3) for initial conditions $(1/3, 1/3)$ and $(1/2, 1/5)$. Do this for parameter values η, $\rho = 1$, 0.3 and also for values η, $\rho = 1$, 1.1.

Create a phase portrait for the logistic equations using both sets of parameter values. Plot the direction field, all equilibrium points, and both solution orbits on the same plot for each set of parameter values.

A Weight Change Model

The main idea behind weight change is simple. If a person takes in more energy than they expend, they gain weight. If they take in less than they expend, they lose weight. Let *energy balance* EB be the difference between *energy intake* EI and *energy expenditure* EE, so that

$$EB = EI - EE.$$

If the balance is positive, weight is gained and similarly if the balance is negative, weight is lost.

A person's body weight at a time t can be expressed as the sum of the weight of their fat tissue $F(t)$ and the weight of their lean tissue $L(t)$; that is, $BW(t) = F(t) + L(t)$. Using this, the change in body weight can be expressed as the following system of ODEs:

$$\begin{aligned} \frac{dF}{dt} &= \frac{(1-p(t))EB(t)}{\rho_F}, \\ \frac{dL}{dt} &= \frac{p(t)EB(t)}{\rho_L}, \end{aligned} \tag{1.4}$$

where $(1 - p(t))$ and $p(t)$ represent the proportion of the energy balance $(EB(t))$ that results in a change in the quantity of fatty or lean tissue, respectively. The constants ρ_F and ρ_L represent the energy density of fatty and lean tissue, approximated as $\rho_F = 9400$ kcal/kg and $\rho_L = 1800$ kcal/kg.

To solve this system, we first need to express $p(t)$ and $EB(t)$ in terms of F and L. These functions will also depend on physical activity level, PAL, and energy intake, EI, which vary among individuals.

We will find an expression for $p(t)$ using Forbes' Law[1] which states that

$$\frac{dF}{dL} = \frac{F}{10.4}.$$

Notice

$$\frac{F}{10.4} = \frac{dF}{dL} = \frac{dF/dt}{dL/dt} = \frac{\dfrac{(1-p(t))EB(t)}{\rho_F}}{\dfrac{p(t)EB(t)}{\rho_L}} = \frac{\rho_L}{\rho_F}\frac{1-p(t)}{p(t)}.$$

Solving for $p(t)$ gives Forbes' equation

$$p(t) = \frac{C}{C + F(t)} \quad \text{where} \quad C = 10.4\frac{\rho_L}{\rho_F}. \tag{1.5}$$

We will now find an expression for $EB(t)$. Recall $EB(t) = EI - EE$. We will use the following expression for energy expenditure (EE) to define $EB(t)$.

$$EE = PAL \times RMR \tag{1.6}$$

1.40–1.69	People who are sedentary and do not exercise regularly, spend most of their time sitting, standing, with little body displacement
1.70–1.99	People who are active, with frequent body displacement throughout the day or who exercise frequently
2.00–2.40	People who engage regularly in strenuous work or exercise for several hours each day

Table 1.1: This is a rough guide for physical activity level (PAL).

where PAL is physical activity level (as previously mentioned) and RMR is resting metabolic rate. Physical activity level can be determined using the table above.

We will use the following equation for computing RMR,

$$RMR = K + \gamma_F F(t) + \gamma_L L(t) + \eta_F \frac{dF}{dt} + \eta_L \frac{dL}{dt} + \beta_{at} EI, \tag{1.7}$$

where $\gamma_F = 3.2$ kcal/kg/d, $\gamma_L = 22$ kcal/kg/d, $\eta_F = 180$ kcal/kg, and $\eta_L = 230$ kcal/kg [2] [3]. Further, we let $\beta_{at} = 0.14$ denote the coefficient for adaptive thermogenesis. Finally, we remark that the constant K can be tuned to an individual's body type directly through RMR and fat measurement, and is assumed to remain constant over time.

Thus, since the input EI is assumed to be known, we can use (1.6), (1.7) and (1.5) to write (1.4) in terms of F and L, thus allowing us to close the system of ODEs.

Specifically, we have

$$\begin{aligned}
RMR = \frac{EE}{PAL} &= K + \gamma_F F(t) + \gamma_L L(t) + \eta_F \frac{dF}{dt} + \eta_L \frac{dL}{dt} + \beta_{at} EI \\
\frac{1}{PAL}(EE - EI + EI) &= K + \gamma_F F(t) + \gamma_L L(t) \\
&\quad + \left(\frac{\eta_F}{\rho_F}(1 - p(t)) + \frac{\eta_L}{\rho_L} p(t)\right) EB(t) + \beta_{at} EI. \\
\left(\frac{1}{PAL} - \beta_{at}\right) EI &= K + \gamma_F F(t) + \gamma_L L(t) \\
&\quad + \left(\frac{\eta_F}{\rho_F}(1 - p(t)) + \frac{\eta_L}{\rho_L} p(t) + \frac{1}{PAL}\right) EB(t).
\end{aligned}$$

Solving for $EB(t)$ in the last equation yields

$$EB(t) = \frac{\left(\frac{1}{PAL} - \beta_{at}\right) EI - K - \gamma_F F(t) - \gamma_L L(t)}{\frac{\eta_F}{\rho_F}(1 - p(t)) + \frac{\eta_L}{\rho_L} p(t) + \frac{1}{PAL}}. \tag{1.8}$$

In equilibrium ($EB = 0$), this gives us

$$K = \left(\frac{1}{PAL} - \beta_{at}\right) EI - \gamma_F F - \gamma_L L. \tag{1.9}$$

Thus, for a subject who has maintained the same weight for a while, one can determine K by using (1.9), if they know their average caloric intake and amount of fat (assume $L = BW - F$).

[1] *Lean body mass-body fat interrelationships in humans*, Forbes, G.B.; *Nutrition reviews*, pgs 225-231, 1987.

[2] *Modeling weight-loss maintenance to help prevent body weight regain*; Hall, K.D. and Jordan, P.N.; *The American journal of clinical nutrition*, pg 1495, 2008

[3] *Quantification of the effect of energy imbalance on bodyweight*; Hall, K.D. et al.; *The Lancet*, pgs 826-837, 2011

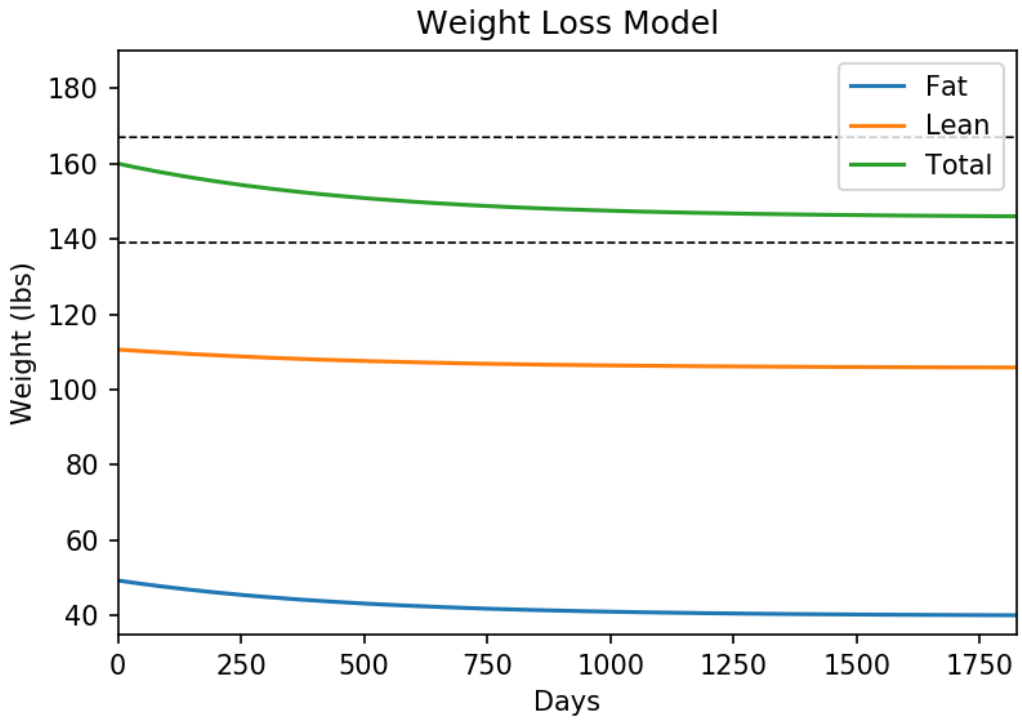

Figure 1.3: The solution of the weight change model for problem 6.

Problem 5. Write the functions `forbes()` which takes as input $F(t)$ and returns Forbe's equation given in (1.5). Also write the function `energy_balance()` which takes as input $F(t)$, $L(t)$, PAL, and EI and returns the energy balance as given in (1.8). Use $\rho_F = 9400$, $\rho_L = 1800$, $\gamma_F = 3.2$, $\gamma_L = 22$, $\eta_F = 180$, $\eta_L = 230$, and $\beta_{AT} = 0.14$.

Using `forbes()` and `energy_balance()`, define the function `weight_odesystem()` which takes as input the current fat and lean weights as an array y and the current time as a float t and returns the right hand side of (1.4) as a tuple.

Problem 6. Consider the initial value problem corresponing to (1.4).

$$\begin{aligned} \frac{dF(t)}{dt} &= \frac{(1-p(t))EB(t)}{\rho_F}, \\ \frac{dL(t)}{dt} &= \frac{p(t)EB(t)}{\rho_L}, \\ F(0) &= F_0, \\ L(0) &= L_0. \end{aligned} \tag{1.10}$$

The following function returns the fat mass of an individual based on body weight (kg), age (years), height (meters), and sex. Use this function to define initial conditions F_0 and L_0 for the IVP above: $F_0 = fat\_mass(args^*)$, $L_0 = BW - F_0$.

```
def fat_mass(BW, age, H, sex):
    BMI = BW / H**2.
    if sex == 'male':
        return BW * (-103.91 + 37.31 * log(BMI) + 0.14 * age) / 100
    else:
        return BW * (-102.01 + 39.96 * log(BMI) + 0.14 * age) / 100
```

Suppose a 38 year old female, standing 5'8" and weighing 160 lbs, reduces her intake from 2143 to 2025 calories/day, and increases her physical activity from little to no exercise (PAL=1.4) to exercising to 2-3 days per week (PAL=1.5).

Use (1.9) and the original intake and phyical activity levels to compute K for this system. Then use `odeint` to solve the IVP. Graph the solution curve for this single-stage weightloss intervention over a period of 5 years. Your plot should match figure 1.3.

Note the provided code requires quantities in metric units (kilograms, meters, days) while our graph is converted to units of pounds and days.

Problem 7. Modify the preceding problem to handle a two stage weightloss intervention: Suppose for the first 16 weeks intake is reduced from 2143 to 1600 calories/day and physical activity is increased from little to no exercise (PAL=1.4) to an hour of exercise 5 days per week (PAL=1.7). The following 16 weeks intake is increased from 1600 to 2025 calories/day, and exercise is limited to only 2-3 days per week (PAL=1.5).

You will need to recompute F_0, and L_0 at the end of the first 16 weeks, but K will stay the same. Find and graph the solution curve over the 32 week period.