17 Transit time crossing a river

Lab Objective: *This lab discusses a classical calculus of variations problem: how is a river to be crossed in the shortest possible time? We will look at a numerical solution using the pseudospectral method.*

Suppose a boat is to be rowed across a river, from a point A on one side of a river ($x = -1$), to a point B on the other side ($x = 1$). Assuming the boat moves at a constant speed 1 relative to the current, how must the boat be steered to minimize the time required to cross the river?

Let us consider a typical trajectory for the boat as it crosses the river. If T is the time required to cross the river, then the position s of the boat at time t is

$$\begin{aligned} s(t) &= \langle x(t), y(t)\rangle, \quad t \in [0, T], \\ s'(t) &= \langle x'(t), y'(t)\rangle, \\ &= \langle \cos\theta(x(t)), \sin\theta(x(t))\rangle + \langle 0, c(x(t))\rangle. \end{aligned}$$

Here $\langle \cos\theta, \sin\theta\rangle$ represents the motion of the boat due to the rower, and $\langle 0, c\rangle$ is the motion of the boat due to the current.

We can relate the angle at which the boat is steered to the graph of its trajectory by noting that

$$\begin{aligned} y'(x) &= \frac{y'(t)}{x'(t)}, \\ &= \frac{\sin\theta + c}{\cos\theta}, \\ &= c\sec\theta + \tan\theta. \end{aligned} \tag{17.1}$$

The time T required to cross the river is given by

$$\begin{aligned} T &= \int_{-1}^{1} t'(x)\, dx, \\ &= \int_{-1}^{1} \frac{1}{x'(t)}\, dx \\ &= \int_{-1}^{1} \sec\theta(x)\, dx. \end{aligned} \tag{17.2}$$

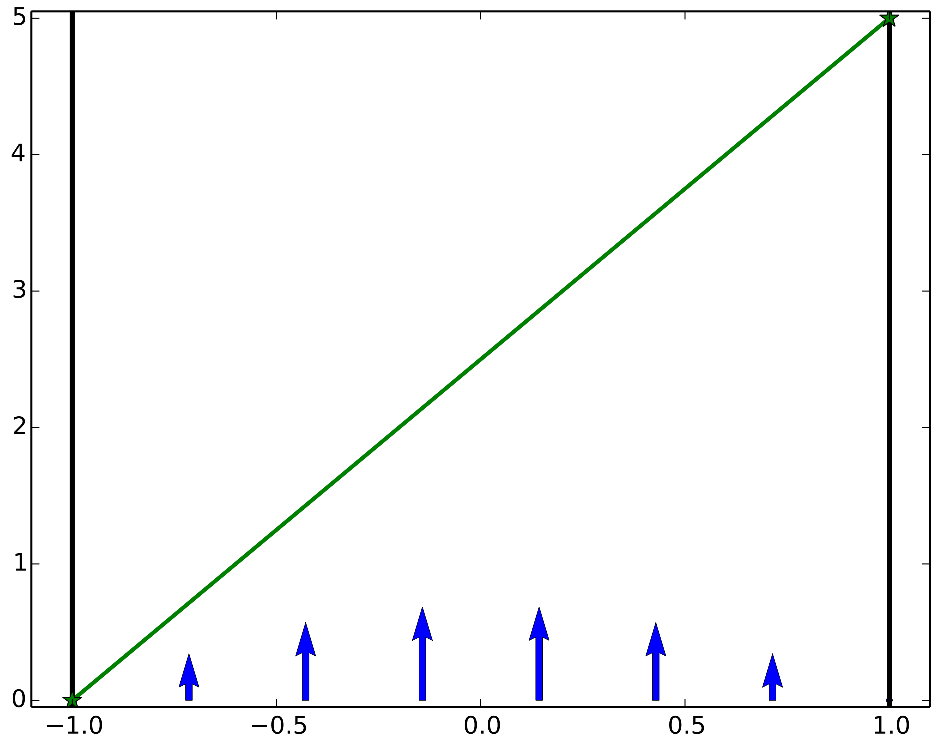

Figure 17.1: The river's current, along with a possible trajectory for the boat.

We would like to find an expression for the total time T required to cross the river from A to B, in terms of the graph of the boat's trajectory. To derive the functional $T[y]$, we note that

$$\begin{aligned} T[y] &= \int_{-1}^{1} \sec\theta \, dx, \\ &= \int_{-1}^{1} \frac{1}{1-c^2}(c\tan\theta + \sec\theta - c^2\sec\theta - c\tan\theta)\, dx, \\ &= \int_{-1}^{1} \frac{1}{1-c^2}(c\tan\theta + \sec\theta - cy')\, dx. \end{aligned}$$

Since

$$\begin{aligned} c\tan\theta + \sec\theta &= \sqrt{1-c^2+(c\sec\theta+\tan\theta)^2}, \\ &= \sqrt{1-c^2+(y')^2}, \end{aligned}$$

we obtain at last

$$T[y] = \int_{-1}^{1} \left[\alpha(x)\sqrt{1+(\alpha y')^2(x)} - (\alpha^2 c y')(x)\right] \, dx, \tag{17.3}$$

where $\alpha = (1-c^2)^{-1/2}$.

Problem 1. Assume that the current is given by $c(x) = -\frac{7}{10}(x^2 - 1)$. (This function assumes, for example, that the current is faster near the center of the river.) Write a Python function that accepts as arguments a function y, its derivative y', and an x-value, and returns $L(x, y(x), y'(x))$ (where $T[y] = \int_{-1}^{1} L(x, y(x), y'(x))$). Use that function to define a second function that numerically computes $T[y]$ for a given path $y(x)$.

Problem 2. Let $y(x)$ be the straight-line path between $A = (-1, 0)$ and $B = (1, 5)$. Numerically calculate $T[y]$ to get an upper bound on the minimum time required to cross from A to B. Using (17.2), find a lower bound on the minimum time required to cross.

We look for the path $y(x)$ that minimizes the time required for the boat to cross the river, so that the function T is minimized. From the calculus of variations we know that a smooth path $y(x)$ minimizes T only if the Euler-Lagrange equation is satisfied. Recall that the Euler-Lagrange equation is

$$L_y - \frac{d}{dx} L_{y'} = 0.$$

Since $L_y = 0$, we see that the shortest time trajectory satisfies

$$\frac{d}{dx} L_{y'} = \frac{d}{dx} \left(\alpha^3(x) y'(x) (1 + (\alpha y')^2(x))^{-1/2} - \alpha^2(x) c \right) = 0. \tag{17.4}$$

Problem 3. Numerically solve the Euler-Lagrange equation (17.4), using $c(x) = -\frac{7}{10}(x^2 - 1)$ and $\alpha = (1 - c^2)^{-1/2}$, and $y(-1) = 0$, $y(1) = 5$.

Hint: Since this boundary value problem is defined over the doimain $[-1, 1]$, it is easy to solve using the pseudospectral method. Begin by replacing each $\frac{d}{dx}$ with the pseudospectral differentiation matrix D. Then impose the boundary conditions and solve.

Problem 4. Plot the angle at which the boat should be pointed at each x-coordinate. (Hint: Use Equation (17.1); see Figure 17.3. Note that the angle the boat should be steered is *not* described by the tangent vector to the trajectory.)

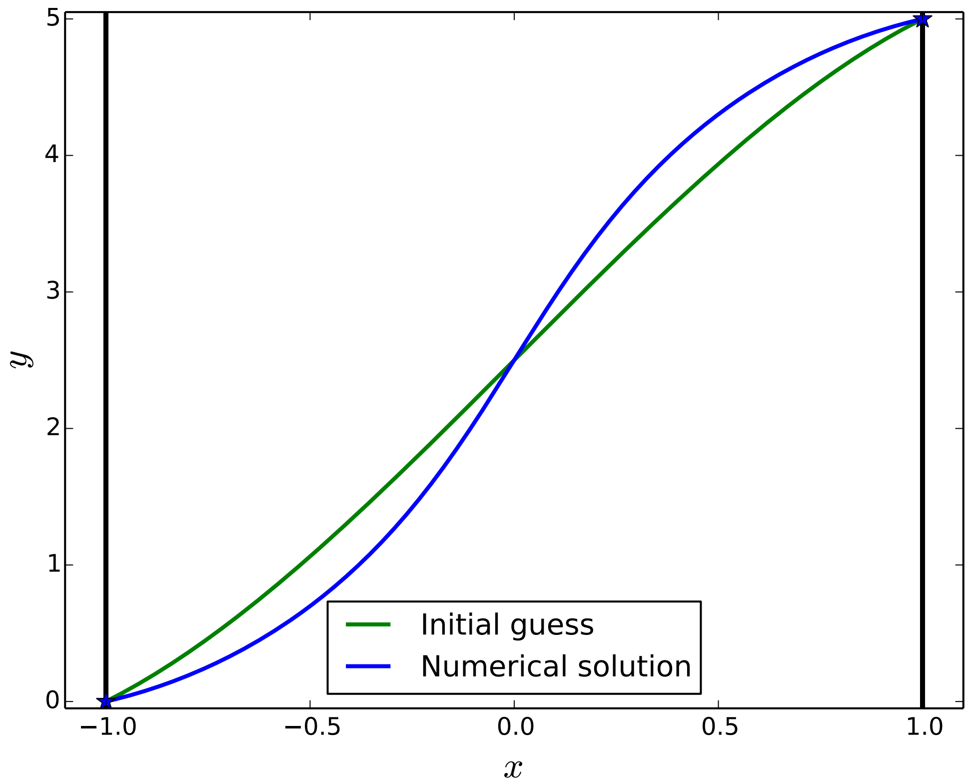

Figure 17.2: Numerical computation of the trajectory with the shortest transit time.

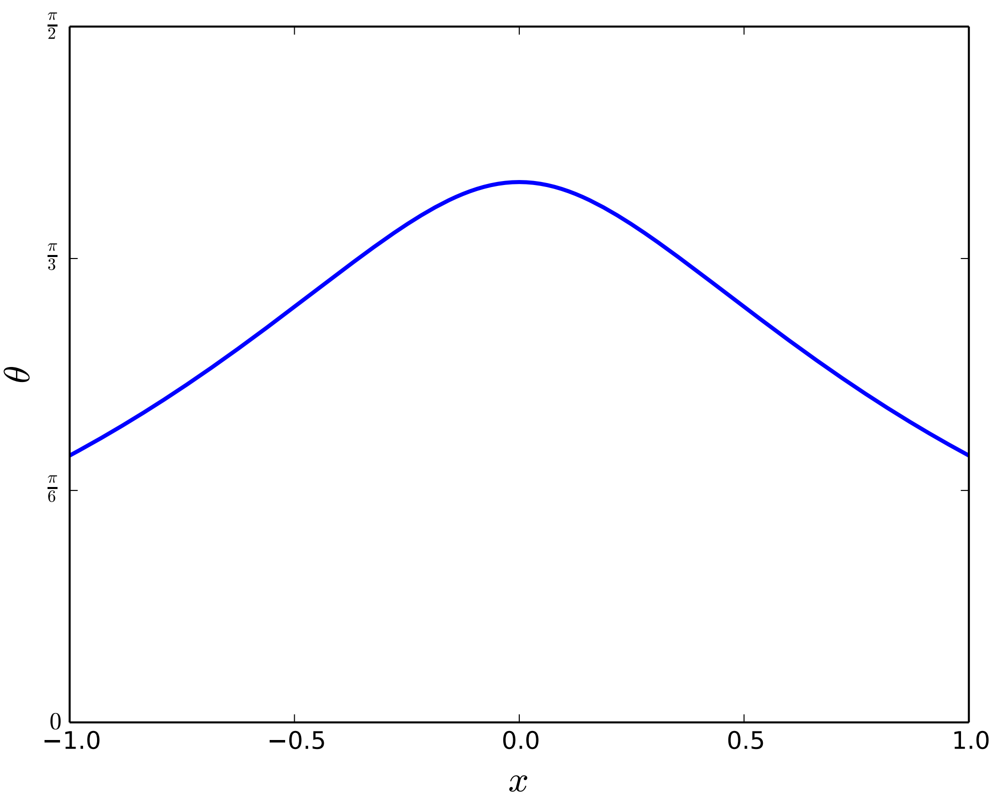

Figure 17.3: The optimal angle to steer the boat.