Lab 4

Modelling the spread of an epidemic: SIR models

The SIR model describes the spread of an epidemic through a large population. It does this by describing the movement of the population through three phases of the disease: those individuals who are *susceptible*, those who are *infectious*, and those who have been *removed* from the disease. Those individuals in the removed class have either died, or have recovered from the disease and are now immune to it. If the outbreak occurs over a short period of time, we may reasonably assume that the total population is fixed, so that $S'(t) + I'(t) + R'(t) = 0$. We may also assume that $S(t) + I(t) + R(t) = 1$, so that $S(t)$ represents the *fraction* of the population that is susceptible, etc.

Individuals may move from one class to another as described by the flow

$$S \to I \to R.$$

Let us consider the transition rate between S and I. Let β represent the average number of contacts made per day that could spread the disease. The proportion of these contacts that are with a susceptible individual is $S(t)$. Thus, one infectious individual will on average infect $\beta S(t)$ others per day. Let N represent the total population size. Then we obtain the differential equation

$$\frac{d}{dt}(S(t)N) = -\beta S(t)(I(t)N)$$

Now consider the transition rate between I and R. We assume that there is a fixed proportion γ of the infectious group who will recover on a given day, so that

$$\frac{d}{dt}R(t) = -\gamma I(t).$$

Note that γ is the reciprocal of the average length of time spent in the infectious phase.

Since the derivatives sum to 0, we have $I'(t) = -S'(t) - R'(t)$, so the differential equations are given by

$$\begin{aligned}\frac{dS}{dt} &= -\beta IS,\\ \frac{dI}{dt} &= \beta IS - \gamma I,\\ \frac{dR}{dt} &= \gamma I.\end{aligned}$$

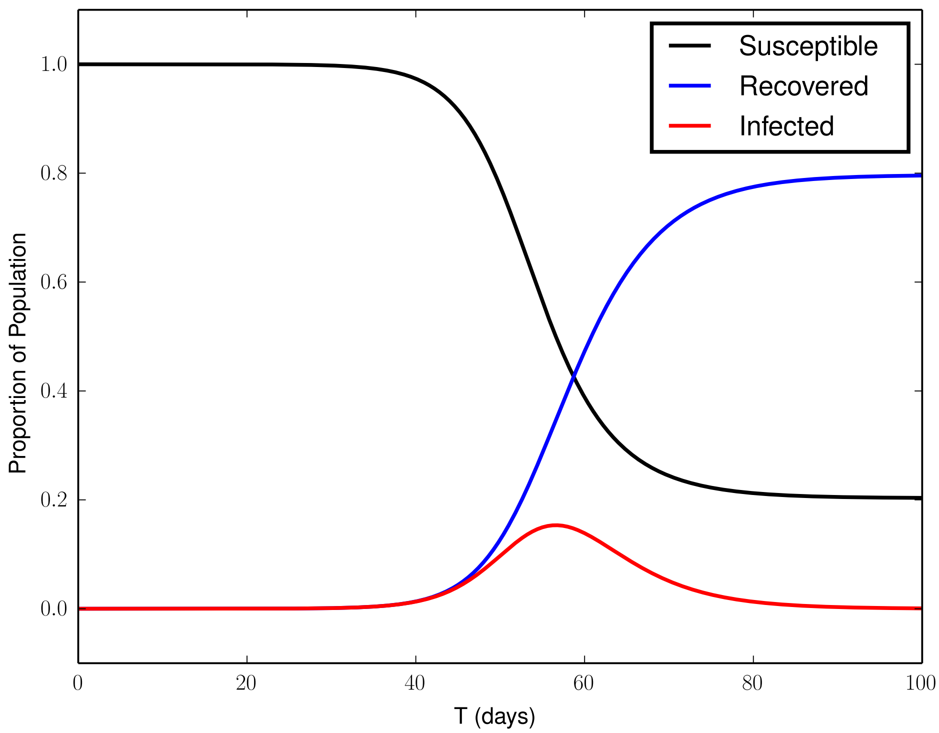

Figure 4.1: Solution to Problem (1)

Problem 1. Solve the IVP

$$\begin{aligned}\frac{dS}{dt} &= -\frac{1}{2}IS,\\ \frac{dI}{dt} &= \frac{1}{2}IS - \frac{1}{4}I,\\ \frac{dR}{dt} &= \frac{1}{4}I,\\ S(0) &= 1 - 6.25 \cdot 10^{-7},\\ I(0) &= 6.25 \cdot 10^{-7},\\ R(0) &= 0,\end{aligned}$$

on the interval $[0, 100]$, and plot your results. See Figure 4.1.

Problem 2. Suppose that, in a city of approximately three million, five have recently entered the city carrying a certain disease. (Suppose they have just entered the infectious state.)

Each of those individuals has a contact each day that could spread the disease, and an average of three days is spent in the infectious state. Find the solution of the corresponding SIR equations for the next fifty days.

At the peak of the infection, how many in the city will still be able to work? (Assume for simplicity that those who are in the infectious state either cannot go to work or are unproductive, etc.) Answer the same question if instead of three days, an average of seven days is spent in the infectious state.

Hint: Find the t value that maximizes I. Then $(S + R) * 3000000$ is the number of individuals who can work at the peak of the infection.

Problem 3. Suppose that, in a city of approximately three million, five have recently entered the city carrying a certain disease. (Suppose they have just entered the infectious state.)

Each of those individuals will make three contacts every ten days that could spread the disease, and an average of four days is spent in the infectious state. Find the solution of the corresponding SIR equations and plot your results. See Figure 4.2.

Variations on the SIR Model

SIS Models describe diseases where individuals who have recovered from the disease do not gain any lasting immunity. There are only two compartments in this model: those who are *susceptible*, and those who are *infectious*.

The basic equations are given by

$$\frac{dS}{dt} = -\beta IS + \gamma I,$$
$$\frac{dI}{dt} = \beta IS - \gamma I$$

If we add to our basic SIR model to account for the death rate and an equal

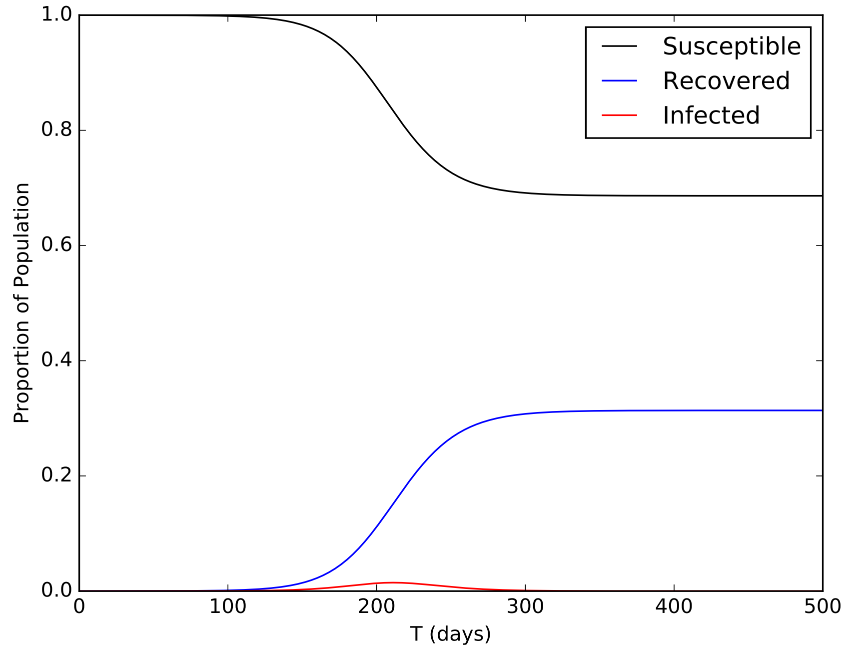

Figure 4.2: Solution to Problem (3).

birth rate, the equations become

$$\begin{aligned}\frac{dS}{dt} &= \mu(1-S) - \beta IS,\\ \frac{dI}{dt} &= \beta IS - (\gamma+\mu)I,\\ \frac{dR}{dt} &= \gamma I - \mu R\end{aligned}$$

SIRS models take the previous model and allow the transfer of individuals from the recovered/removed class to rejoin the susceptible class.

$$\begin{aligned}\frac{dS}{dt} &= fR + \mu(1-S) - \beta IS,\\ \frac{dI}{dt} &= \beta IS - (\gamma+\mu)I,\\ \frac{dR}{dt} &= -fR + \gamma I - \mu R.\end{aligned}$$

The next exercise uses a variation of the basic SIR model to describe the spread of measles. It assumes that the rate at which measles is contracted depends on the season, i.e. the rate is periodic. That allows us to formulate the yearly occurrence rate for measles as a boundary value problem. To solve this problem we will use a full-featured BVP solver that is available as a Python package. Several industrial-grade BVP solvers have been written in Fortran. One of these, `bvp_solver`, has been wrapped for Python and is available as a scikit. If you have not installed it, you can install it by running the command `pip install scikits.bvp_solver` in the command line. The code below demonstrates how to use `bvp_solver` to solve the BVP

$$\epsilon y'' + yy' - y = 0, \quad y(-1) = 1, \quad y(1) = -1/3.$$

```
import numpy as np
from scikits import bvp_solver
import matplotlib.pyplot as plt

epsilon, lbc, rbc = .1, 1., - 1. / 3.

def ode(x , y):
    return np.array([y[1] , (1. / epsilon) * (y[0] - y[0] * y[1])])

# The BVP solver package expects you to pass it the boundary
# conditions as a callable function that computes the difference
# between a guess at the boundary conditions
# and the desired boundary conditions.
# When we use the BVP solver, we will tell it how many constraints
# there should be on each side of the domain so that it knows
# how many entries to expect in the tuples BCa and BCb.
# In this case, we have one boundary condition on either side.
# These constraints are expected to evaluate to 0 when the
# boundary condition is satisfied.
def bcs(ya, yb):
    BCa = np.array([ya[0] - lbc])   # 1 Boundary condition on the left
    BCb = np.array([yb[0] - rbc])   # 1 Boundary condition on the right
    return BCa, BCb

problem = bvp_solver.ProblemDefinition(num_ODE=2,
                                       num_parameters=0,
                                       num_left_boundary_conditions=1,
                                       boundary_points=(-1, 1),
                                       function=ode,
                                       boundary_conditions=bcs)

solution = bvp_solver.solve(problem, solution_guess=(- 1. / .3, - 4. / 3.))

A = np.linspace(-1., 1., 200)
T = solution(A)
plt.plot(A, T[0,:], '-k', linewidth=2.)
plt.show()
```

Problem 4. SEIR models are another variation of the basic SIR model. Basically they added another compartment, called the *exposed* or *latency* phase, to the basic compartments *susceptible*, *infectious*, and *recovered*.

An SEIR model is used to describe the spread of measles (see [a]). The rate at which susceptible individuals may contract measles is seasonal, and corresponds to a periodic function $\beta(t) = \beta_0(1 + \beta_1 \cos 2\pi t)$. Parameters μ and λ represent the birth rate of the population and the latency period of measles, respectively. η represents the infectious period before an individual moves from the infectious class to the recovered class. After recovery an

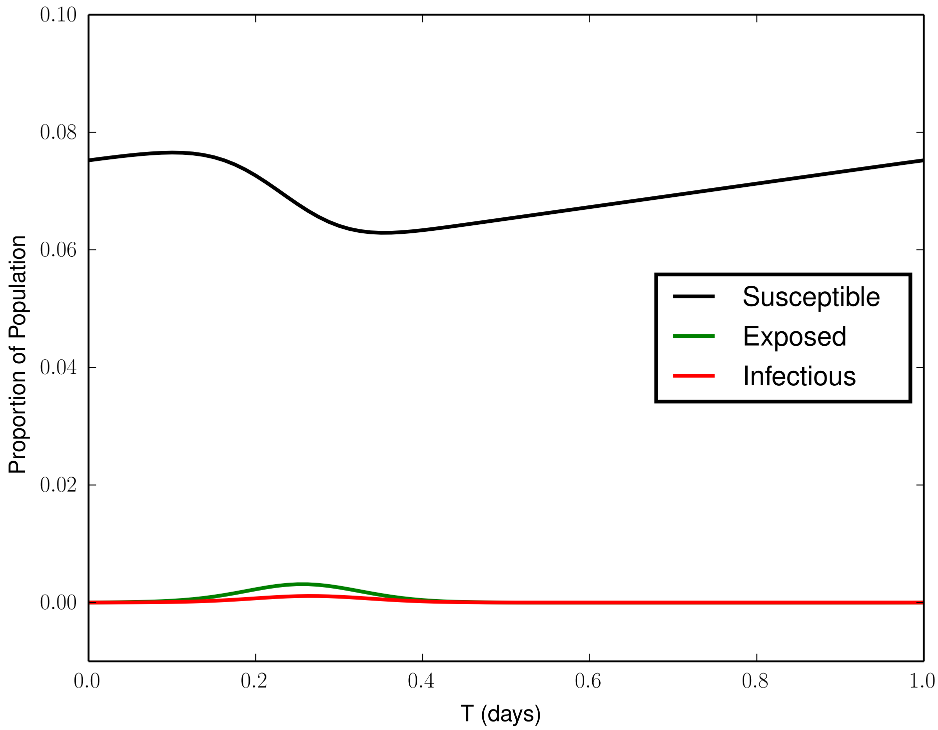

Figure 4.3: Solution to Problem (4)

individual remains immune. The boundary value problem is given by

$$\begin{bmatrix} S \\ E \\ I \end{bmatrix}' = \begin{bmatrix} \mu - \beta(t)SI \\ \beta(t)SI - E/\lambda \\ E/\lambda - I/\eta \end{bmatrix},$$

$$S(0) = S(1),$$
$$E(0) = E(1),$$
$$I(0) = I(1)$$

Solve this BVP with parameters $\beta_1 = 1$, $\beta_0 = 1575$, $\eta = 0.01$, $\lambda = .0279$, and $\mu = .02$. Note: in this case, time is measured in years, so run the solution over the interval $[0, 1]$ to show a one-year cycle. The boundary conditions are really just saying that the year will begin and end in the same state.

Hint: `bvp_solver` requires *separated boundary conditions*. In other words, each equation in the set of boundary conditions can only include values at one end of the interval. To deal with this, let $C = [C_1, C_2, C_3]$, and add the equation

$$C' = 0$$

to the system of ODEs given above (for a total of 6 equations). Then the boundary conditions can be separated using the following trick:

$$\begin{pmatrix} C_1(0) \\ C_2(0) \\ C_3(0) \end{pmatrix} = \begin{pmatrix} S(0) \\ E(0) \\ I(0) \end{pmatrix}, \quad \begin{pmatrix} C_1(1) \\ C_2(1) \\ C_3(1) \end{pmatrix} = \begin{pmatrix} S(1) \\ E(1) \\ I(1) \end{pmatrix}.$$

Now C_1, C_2, C_3 become the 4th, 5th, and 6th rows of your solution matrix, so the 3 boundary conditions for the left are obtained by subtracting the last three entries of $y(0)$ from the first three entries. Similarly, your right boundary conditions will look like $yb[0:3] - yb[3:]$.

When you code your boundary conditions, note that `bvp_solver` changes the initial conditions to force all the entries in the two arrays to be zero. You can use the initial conditions from Fig. 4.3 as your initial guess (which will be an array of 6 elements). Remember that the initial infected proportion is small, not 0.

[a]Numerical Solution of Boundary Value Problems for Ordinary Differential Equations, by Aescher, Mattheij, and Russell